

On dynamics of multi-phase elastic-plastic media

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Abstract

The paper is concerned with dynamics of multi-phase media consisting of a solid permeable material and a compressible Newtonian fluid. Governing macroscopic equations are derived starting from the space-averaged microscopic mass and momentum balances. The Reynolds stress models (i.e., momentum dispersive fluxes) are discussed, and a suitable model is developed. In the case of granular media the solid constituent is considered as an elastic-plastic matrix, and the yield condition is approximated by Coulomb friction law. It is revealed that the classical principle of maximum plastic work is not, in general, valid for granular media, and an appropriate variational principle is developed. This novel principle coincides with the maximum plastic work principle for the case of cohesionless granular media.

1 Introduction

Multi-phase mixtures play a vital part in many natural phenomena and branches of engineering (e.g., [4], [10], [16], [19], [20], [25], [27], [28], [29]), and hence the development of multi-phase dynamics is of great scientific and industrial importance. We restrict our consideration to isotropic (e.g., [4], [26], [32]) permeable (granular, porous, etc.) media consisting of a solid matrix and a compressible Newtonian fluid. Such media have received the most study (see, e.g., [1], [2], [4], [12], [27], [28], [29], [33], [37]), nevertheless a number of important problems still remain to be solved. This paper is mainly concerned with the two important problems, namely, modeling of dispersive flux of momentum [4] as well as development of a variational principle for plastic deformations [15] of fluid-saturated granular media.

The so-called macroscopic balance equations are mainly developed by a method of averaging (see, e.g., [4], [11], [27], [28]) of micro-equations. The method of averaging over elementary volume of a multi-phase medium containing the full ensemble of realizations was originally suggested by Nikolaevskiy

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et al. (1970) (see the references in [27], [28]). This approach, in contrast to the purely phenomenological one (e.g., [16], [28]) gives a possibility to evaluate theoretically the type of constitutive laws and sometimes the values of rheological parameters [27]. However, because of non-linearity, the averaging procedure leads to the macro-equations that are similar to Reynolds equations (e.g., [3], [23]) for turbulent flows, and hence the problem of Reynolds stress (i.e., dispersive flux of momentum [4]) modeling is coming into play. In the fluid phase the pseudo-Brownian velocity pulsations exist even under small Reynolds number as a result of chaotic micro-structure of multi-phase media, and these pulsations are mainly changing in space [28]. For such a motion the dispersivity tensor [32] in the developed models of convective diffusion in, e.g., porous media [27, sec. 7.3] depends on the average velocity, whereas, because of Galilei-Newton principle of relativity [38, V. 2, p. 105], the turbulence model parameters do not depend on the velocity as such (see, e.g., [23], [3]). Hence, the models of turbulent diffusion, as they exist, are not, in general, applicable to multi-phase media. In the case of granular media the Reynolds stresses (i.e., momentum dispersive fluxes) are often believed as negligible [27]. However, in developing models for correct simulation of processes accompanying, e.g., fluid flow in vicinities of gas wells having high production, underground explosion works, oil and water wells rehabilitation and stimulation by shock technologies (steam injection, aggressive pressure pulsing, etc.), meteorological flows over urban and vegetative canopies, and so on, the momentum dispersive fluxes must be taken into consideration (e.g., [4], [10], [18], [28]).

A dispersive flux model of an extensive quantity was suggested in [4] for a microscopically laminar flow regime. The model is essentially based on the modified rule [4, Eq. 2.3.48] for volume averaging of a spatial derivative. Since the modified averaging rule is also used in the development of the total viscous resistance expression as well as the macroscopic momentum balance equation for a fluid phase [4, sec. 2.6], we consider the applicability of this rule to a viscous fluid flow through a permeable medium.

The authors [4] attempted to develop the modified rule for a quantity, G , that attains no maximum or minimum value within the void space of a representative elementary volume (REV). On this basis it is assumed [4, p. 125] that the quantity G is a harmonic function on the microscopic level. This demand is sufficient but not necessary condition for the validity of the maximum principle, and hence it could be too restrictive. To demonstrate it we assume, for the sake of simplicity, that the solid matrix is immobile and the entire void space is occupied by a single Newtonian fluid with the density $\rho = \text{const}$ and the dynamic viscosity $\mu = \text{const}$. Eliminating the body force of gravity by subtraction from the true pressure p of the hydrostatic pressure, the Navier-Stokes and continuity equations (e.g., [3], [23], [35]) can be written in the following non-dimensional form:

$$S_h \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -E_u \nabla P + \frac{1}{Re} \nabla^2 \mathbf{V}, \quad (1)$$

$$\nabla \cdot \mathbf{V} = 0, \quad S_h \equiv \frac{l_*}{V_* t_*}, \quad E_u \equiv \frac{P_*}{\rho V_*^2}, \quad Re \equiv \frac{V_* l_*}{\nu}, \quad (2)$$

where \mathbf{V} , P are the non-dimensional velocity and pressure, respectively; ν denotes the kinematic viscosity; S_h , E_u , Re are, respectively, Struhla, Euler, and Reynolds numbers; the reference quantities are denoted by an asterisk. Multiplying (1) by ∇ we obtain, in view of the first equation in (2), that

$$\nabla^2 P = -\frac{1}{E_u} \nabla \cdot [(\mathbf{V} \cdot \nabla) \mathbf{V}]. \quad (3)$$

Thus, the basic equality, i.e., the Laplace equation for the pressure [4, Eq. 2.6.6], can be approximately valid if the right-hand side in (3) is negligible, i.e., in general, if $E_u \gg 1$. Hence, we conclude that the derivation of the viscous resistance expression as well as the macroscopic momentum balance equation [4, sec. 2.6], relying on the modified averaging rule, can be, in general, valid for a creeping flow only.

The applicability of the modified averaging rule for the development of the momentum dispersive flux model is based on the assumption that the linear momentum density ($\rho \mathbf{V}$) is a harmonic function on the microscopic level. In view of (1) we obtain that $\rho \mathbf{V}$ will be an approximate solution of the Laplace equation if the following inequalities will be valid simultaneously: $S_h Re \ll 1$, $Re \ll 1$, $E_u Re \ll 1$. In general, it is possible for a very specific flow. As an example, let us consider the Newtonian fluid flow (see the governing equations in (1) and (2)) through a porous medium as made up of a bundle of parallel tubes whose radii are assumed to be uniform in size. In the case of a steady-state laminar flow regime we have (in every tube), in fact, Hagen-Poiseuille flow [35, p. 117]. In polar coordinates the velocity (V) distribution over a cross-section of a tube can be written in the form

$$V = \frac{\zeta R^2}{4\mu} \left(1 - \frac{r^2}{R^2}\right), \quad 0 \leq r \leq R, \quad \zeta = \frac{P_0 - P_L}{L}, \quad (4)$$

where R denotes the radius of the tube; L denotes the full length of the tube; ζ denotes the pressure drop; P_0 , P_L are the pressures at the bases of the tube. Obviously, the linear momentum density (ρV) in the case of Hagen-Poiseuille flow is not a harmonic function. To elucidate the conditions wherein ρV is an approximate solution of the Laplace equation we assume that the reference velocity is equal to the maximum value of the velocity in Hagen-Poiseuille flow (i.e., $V_* = 0.25\zeta R^2/\mu$), $l_* = R$, and $P_* = P_0 - P_L$. In such a case we obtain that $E_u Re = 4L/R$, and hence ρV will be an approximate solution of the Laplace equation if $R \gg L$. The last inequality is to say that the influence of the boundary (non-slip) conditions on the flow regime must be negligible. This is by no means the case of porous media. Thus, we conclude that the applicability of the modified averaging rule to the linear momentum density ($\rho \mathbf{V}$) is, in general, questionable.

Using the modified averaging rule as well as a number of additional assumptions and approximations, Bear and Bachmat [4] found that the dispersive flux of an extensive quantity is proportional to the gradient of the mean density of the extensive quantity. Hence, the momentum dispersive flux is proportional to

the gradient of the mean momentum density of mass. The mean momentum density in its turn may be decomposed into two fluxes: a macroscopic advective flux and a dispersive flux of mass. The latter flux, in view of the model by Bear and Bachmat [4], is proportional to the gradient of the mean density of mass. Thus, following [4], we obtain an interesting result that the macroscopic momentum balance equation is a third-order partial differential equation. However, in view of the foregoing analysis, the basis for this result is not convincing. Two more points need to be made. The momentum dispersive flux (see, e.g., left-hand side in [4, Eq. 2.6.52]), which is a symmetrical tensor, is approximated by the tensor (see the right-hand side in [4, Eq. 2.6.52]), which is not, in general, symmetric. Furthermore, the kinetic energy of dispersion (analog to the kinetic energy of turbulence, see [3, p. 221]) is not taken into consideration in the momentum dispersive flux model in [4].

Let us note, however, that the approach employed by Bear and Bachmat [4] deserves more attention. This approach and the conclusions, as it is underlined by the authors, are equally valid for any extensive quantity. It is a reflection of the plausible assumption that the mechanisms of heat, mass, and momentum transfer in multi-phase media are identical. In the semi-empirical theory of turbulence the similar assumption is called as Reynolds analogy (see, e.g., [3], [23], [35]). Specifically, Anderson et al. [3] pointed out that the ratio of the diffusivities for the turbulent transport of heat and momentum (turbulent Prandtl number) is a well-behaved function across the flow, and the Prandtl number varies between about 0.6 at the outer edge of the boundary layer to about 1.5 near the wall. Assuming that this assumption is valid, we can conclude [4] that the coefficient of mechanical dispersion (dispersivity tensor, see [32]) has analogous form for any conservative extensive quantity, be it heat, mass, or linear momentum.

It should also be remarked that Bear and Bachmat [4] suggested the totally irreversible model of the momentum dispersive flux for the case of microscopically laminar flow regime provided the density is constant. In contrast to [4], a reversible model of the Reynolds stress tensor (i.e., the momentum dispersive flux) is developed by Nikolaevskiy [28]. It is assumed that the local velocity (\mathbf{V}) is a stochastic function of the average velocity ($\bar{\mathbf{V}}$). In such a case the vector of mean velocity is transformed randomly (see [26], [28]) into the velocity pulsations ($\mathbf{V}' \equiv \mathbf{V} - \bar{\mathbf{V}}$). This stochastic transformation is represented, in both the symbolic and indicial notations, as follows:

$$\mathbf{V}' = \mathbf{L} \cdot \bar{\mathbf{V}}; \quad V'_i = L_{ij} \bar{V}_j, \quad (5)$$

where the repeated indices, as usual, denote summation; the tensor \mathbf{L} is determined by the structure of porous medium, Re , and a realization parameter, corresponding to the random character of the medium. Then, the equations for the Reynolds stresses are, in fact, written [28, sec. 4.4.1] in the following form

$$\overline{\mathbf{V}'\mathbf{V}'} = \mathbf{T} : (\bar{\mathbf{V}}\bar{\mathbf{V}}); \quad \overline{V'_i V'_j} = T_{ijkl} \bar{V}_k \bar{V}_l, \quad T_{ijkl} = \overline{L_{ik} L_{jl}}, \quad (6)$$

To estimate the kinetic energy of dispersion, $0.5 \overline{V'_i V'_i}$, we assume, in view of the isotropy of the porous medium [26], that the second rank tensor with the

components $\overline{L_{ik}L_{il}}$ will be isotropic. Then, by virtue of (6), we obtain

$$\overline{(\mathbf{V}')^2} \equiv \overline{V'_i V'_i} = \text{tr} \overline{V'_i V'_j} = \overline{L_{ik}L_{il}} \overline{V_k V_l} = \omega (\overline{\mathbf{V}})^2, \quad \omega = \frac{1}{3} \mathbf{L} : \mathbf{L}. \quad (7)$$

Let us assess the validity of the assumption (5) and, hence, the model (6). Notice, if the direction of $\overline{\mathbf{V}}$ in (5) will be changed to the opposite one (i.e., $\overline{\mathbf{V}} \rightarrow -\overline{\mathbf{V}}$), then the only direction of \mathbf{V}' will be changed (i.e., $\mathbf{V}' \rightarrow -\mathbf{V}'$). However, such a property is not, in general, exhibited by the flow, e.g., in convergent and divergent channels. The exact solution of the Navier-Stokes equations for such a flow was originally found by Jeffery and Hamel (see the brief sketch in [35, pp. 104-106]). The velocity distribution for the convergent and for the divergent channel differ significantly from each other, and in the latter case vary greatly with Reynolds number (see, e.g., [13], [35]). Inasmuch as the flow in such channels is essentially irreversible, we conclude that (5) can, in general, be approximately valid under low Reynolds numbers only.

Frick [13] points out that a symmetric divergent Jeffery-Hamel flow exists only if the Reynolds number $Re < \widehat{Re}$ and the opening angle $\alpha < \widehat{\alpha}$, where the values \widehat{Re} and $\widehat{\alpha}$ meet the following condition: $\widehat{Re} = 6 (\pi^2 / \widehat{\alpha} - \widehat{\alpha})$. Therefore, choosing $\alpha \geq \pi$ we obtain at least one region of back-flow, whichever Re might be, and, resulting from it, separation. Separation of a boundary layer, in reality, gives rise to vortices [35, Sec. 2] resulting in turbulence. For instance, measurements demonstrate [35, Sec. 7.2.6] that the flow of a free jet can be laminar up until about $Re = 30$, where Re is referred to the outlet velocity and the slit height. Hence, in a real granular medium, the solid phase of which composed, in general, of irregular in size and shape grains, the vortices (i.e., micro-vortices) can arise under low Reynolds numbers as the result of steep rise in pressure at sharp edges, fractures, sudden expansions, etc'. The higher Re , the more micro-vortices arise within an REV. Nevertheless, the flow is still laminar. Let us note, however, that the vortices are the main source of turbulence [6]. The turbulent flow stems from the lose of vortex stability and degradation of vortex structure on further rise in Re . If Re is over a critical Reynolds number, then the vortex structure breaks into turbulence within a part of the REV. The higher Re , the most part of the flow will be turbulent. The above speculation is supported by experimental results (see, e.g., [5], [7], [21] and references therein), and hence it might be differentiated the following regimes of flow: 1) Laminar regime, where the resistance to the flow is directly proportional to the mean velocity (Darcy linear law). 2) Laminar regime, where the resistance is nonlinear (Darcy-Hazen-Dupuit-Forchheimer law). 3) Transition regime. 4) Turbulent flow. Thus, from the preceding, it appears that the assumption (5) and, hence, the model (6) can be approximately valid in the case of the first flow regime (laminar regime, Darcy linear flow), otherwise the validity of (5) is questionable. The same conclusion is valid for the model developed by Bear and Bachmat [4].

Recently, several turbulence models have been established for turbulent flows (i.e., for the fourth regime) in permeable (granular, porous, etc'.) media (see, e.g., [10], [18], [22], [24], [31], [39], [40], and references therein). Let us note that the widely used one- and two-equation turbulence models (e.g., [10], [18], [22],

[24], [31], [40]) are only valid in the fully turbulent regime, i.e., these models are not appropriate for the near-wall region [3, pp. 231-233]. Owing to this, the validity of these models for turbulent flows in porous media is questionable. Thus, as already noted in [19, p. 63], modeling of the Reynolds stresses for multi-phase flows is still at its infancy.

The present study (Sec. 2) is devoted to the development of a sufficiently simple mathematical model for simulating flows in permeable media under all regimes. To avoid the specific problems of dispersive flux modeling associated with inconstant phase densities (e.g., [3], [4]), we will develop a Reynolds form of the balance equations in mass-averaged variables [3, p. 201]. Then we will develop a model of the dispersive flux assuming that: (i) the Reynolds stresses are linearly dependent on the mean velocity derivatives (see, e.g., [3], [4], [19], [23], [28]), (ii) the Reynolds stress tensor will be isotropic if the mean velocity is constant [19], (iii) the Reynolds analogy is valid for multi-phase media.

Considering the granular medium as an elastic-plastic one (e.g., [16], [17], [25], [27], [28]), we are concerned with validity of the classical principle of maximum plastic work, as applied to granular media. It can be seen from (e.g., [15, p. 58], [38, V. 4, ch. 12]) that the associative flow rule (the normality law [15, p. 58]) is the necessary condition for validity of the maximum plastic work principle. Nikolaevskiy [27], [28] has revealed that in the case of granular media the irreversible strain-rate must be determined by the non-associative flow rule. Thus, we may conclude that the principle of maximum plastic work [15] is not in general valid for granular media. Importance of variational principles in physics, including the maximum plastic work principle developed by von Mises, Taylor, and Bishop and Hill, is well known (e.g., [14], [15], [34], [34]). In particular, Han and Reddy [15] wrote, "The principle of maximum plastic work is a vital constituent of the theory of plasticity." Moreover, currently, numerical methods for solving the problems of elastic-plastic deformations, including construction of monotone (e.g., [8], [9]) difference schemes, are based on non-classic formulation of variational principles, namely variational inequalities ([15], [30], [34]). Hence, there is a need to develop a proper variational principle for plastic deformation of granular media. With this in mind we will assume the validity of the Coulomb yield condition (e.g., [25], [27], [28]) and the non-associative flow rule (e.g., [27], [28]). Then the desired variational principle will be rigorously deduced on the basis of irreversible thermodynamics (e.g., [14], [27]), as applied to granular media. Let us note that in such a development one would like to use a theoretical premise instead of the Coulomb friction law that is nothing more than an empirical relationship [25]. Currently another approach, free of Coulomb condition, is suggested by Jiang and Liu [17]. The novel elastic theory [17] accounts for mechanical yield by a feature of non-linear elasticity only. However, from physical point of view, such an approach is not well founded, as it is not obvious that solid friction at inter-particle contacts can be totally accounted for by non-linear elasticity. Furthermore, Jiang and Liu [17] developed their theory for the case of cohesionless granular media only. Hence, the use of this theory [17], as it exists, for development of the variational principle is out of question.

2 Momentum transport

We start our investigation with the volume averaged balance equations [4]. In the absence of phase transition, the macroscopic mass and, respectively, momentum balance equations for a fluid phase can be written in the form

$$\frac{\partial}{\partial t} (\phi \overline{\rho_f}) = -\nabla \cdot (\phi \overline{\rho_f \mathbf{V}_f}), \quad (8)$$

$$\begin{aligned} \frac{\partial}{\partial t} (\phi \overline{\rho_f \mathbf{V}_f}) &= -\nabla \cdot \phi (\overline{\rho_f \mathbf{V}_f \mathbf{V}_f} - \overline{\sigma_f}) \\ &+ \frac{\phi}{U_f} \int_{S_{fs}} \sigma_f \cdot \mathbf{n} ds + \phi \overline{\rho_f \mathbf{g}_f}, \end{aligned} \quad (9)$$

where ρ_f denotes the fluid density, ϕ denotes the void fraction, \mathbf{V}_f denotes the velocity of the fluid, $\sigma_f \equiv -p_f \mathbf{I} + \tau_f$ denotes the fluid stress tensor, p_f denotes the pressure, \mathbf{I} denotes the identity tensor, τ_f denotes the viscous stress tensor, \mathbf{g}_f denotes the body force vector, U_f denotes the volume occupied by the fluid phase within an REV, S_{fs} denotes the surface of the fluid-solid interface, \mathbf{n} denotes the outward unit vector to U_f on S_{fs} . Let U_s denote the volume occupied by the solid phase within the REV, and let e_f and e_s denote any variables referred to the fluid and solid phases, respectively. Hereinafter $\overline{e_f}$ and $\overline{e_s}$ denote the volume averages of e_f and e_s over, respectively, U_f and U_s . Let $\tilde{e}_f \equiv (\overline{\rho_f e_f} / \overline{\rho_f})$ and $\tilde{e}_s \equiv (\overline{\rho_s e_s} / \overline{\rho_s})$ denote the mass-averaged variables. Following Nikolaevskiy [27] we write the phase interaction force (\mathbf{F}) in the form

$$\mathbf{F} \equiv \frac{\phi}{U_f} \int_{S_{fs}} \sigma_f \cdot \mathbf{n} ds - \overline{p} \nabla \phi = -\mathbf{R} \cdot \tilde{\mathbf{U}}, \quad \mathbf{U} \equiv \mathbf{V}_f - \tilde{\mathbf{V}}_s, \quad (10)$$

where $\mathbf{R} (= \mu \psi \phi \mathbf{K}^{-1})$ denotes the symmetric resistance tensor [27], \mathbf{K} denotes the tensor of absolute permeability, $\psi = \psi(Re)$, $Re \equiv |\tilde{\mathbf{U}}| l_* / \nu$ denotes the local Reynolds number, $\nu = \mu / \rho_f$ denotes the kinematic viscosity, l_* denotes the length parameter characterizing the void space.

Using the mass-averaged variables and by virtue of (10) we obtain from (8), (9) the following balance equations:

$$\frac{\partial}{\partial t} (\phi \overline{\rho_f}) = -\nabla \cdot (\phi \overline{\rho_f \tilde{\mathbf{V}}_f}), \quad (11)$$

$$\begin{aligned} \phi \overline{\rho_f} \frac{d}{dt} \tilde{\mathbf{V}}_f &= -\phi \nabla \overline{p_f} - \mathbf{R} \cdot (\tilde{\mathbf{V}}_f - \tilde{\mathbf{V}}_s) \\ &+ \phi \overline{\rho_f \mathbf{g}_f} + \nabla \cdot \phi \overline{\tau_f} - \nabla \cdot \phi \overline{\rho_f \mathbf{V}_f'' \mathbf{V}_f''}, \end{aligned} \quad (12)$$

where $e_f'' \equiv e_f - \tilde{e}_f$ denotes the fluctuations of a variable e_f . We will also use $e_f' = e_f - \overline{e_f}$. Since the fluid phase is a Newtonian liquid, we can write (e.g., [4], [38]) that

$$\tau_f = 2\mu \mathbf{S} - \kappa \nabla \cdot \mathbf{V}_f \mathbf{I}, \quad \mathbf{S} = \frac{1}{2} [\nabla \mathbf{V}_f + (\nabla \mathbf{V}_f)^*], \quad (13)$$

where $\kappa (= \frac{2}{3}\mu - \zeta)$ denotes the coefficient of bulk viscosity, ζ denotes the second coefficient of viscosity, $(\)^*$ denotes a conjugate tensor. Estimating the mean value of the viscous tensor ($\overline{\boldsymbol{\tau}_f}$) we note [3] that in practice the viscous terms involving $\widetilde{\mathbf{V}}_f'' \equiv -\overline{\rho_f' \mathbf{V}_f'} / \overline{\rho_f}$ can be neglected, and hence we obtain

$$\overline{\boldsymbol{\tau}_f} \approx 2\overline{\mu}\widetilde{\mathbf{S}} - \overline{\kappa}\nabla \cdot \widetilde{\mathbf{V}}_f \mathbf{I}, \quad \widetilde{\mathbf{S}} = \frac{1}{2} \left[\nabla \widetilde{\mathbf{V}}_f + \left(\nabla \widetilde{\mathbf{V}}_f \right)^* \right]. \quad (14)$$

Notice, the first equality in (14) may be considered as strict equation introducing new characteristics $\overline{\mu}$ and $\overline{\kappa}$ instead of the mean value of conventional coefficients of viscosity. In such a case we obtain that $\overline{\nu} = \overline{\mu} / \overline{\rho_f}$.

The dispersion in the case of isotropic granular media is, in general, non-isotropic [4], which is to say that at every point of the flow there must be defined a symmetric tensor (\mathbf{D}_f) of second rank (e.g., [4], [23], [27], [32]) such that the dispersive flux of the momentum will be dependent on $\mathbf{D}_f \cdot \nabla \widetilde{\mathbf{V}}_f$. Since the Reynolds stresses form a symmetric tensor, it is natural to consider this tensor as a linear function of the following symmetric one

$$\boldsymbol{\Phi}_f \equiv \frac{1}{2} \left[\mathbf{D}_f \cdot \nabla \widetilde{\mathbf{V}}_f + \left(\mathbf{D}_f \cdot \nabla \widetilde{\mathbf{V}}_f \right)^* \right]. \quad (15)$$

We will assume that because of chaotic micro-structure of the isotropic granular medium, the Reynolds stress tensor will be isotropic if $\boldsymbol{\Phi}_f = 0$. Thus, we may write that

$$-\overline{\rho_f} \widetilde{\mathbf{V}}_f'' \widetilde{\mathbf{V}}_f'' = \overline{\rho_f} (\boldsymbol{\Phi}_f + d\mathbf{I}), \quad (16)$$

where d is a scalar parameter. In view of the last equality in (10) we obtain that $\mathbf{V}_f'' = \mathbf{U}''$. Then, assuming that

$$|\widetilde{\mathbf{U}}|^2 \approx (1 + \omega_f) |\widetilde{\mathbf{U}}|^2, \quad (17)$$

where $\omega_f (\geq 0)$ is a new non-dimensional parameter, we can write

$$\begin{aligned} \left(\widetilde{\mathbf{V}}_f'' \right)^2 &= \widetilde{\mathbf{U}} \cdot \widetilde{\mathbf{U}} - 2\widetilde{\mathbf{U}} \cdot \widetilde{\mathbf{U}} + \widetilde{\mathbf{U}} \cdot \widetilde{\mathbf{U}} = \left(\widetilde{\mathbf{U}} \right)^2 - \left(\widetilde{\mathbf{U}} \right)^2 \approx \\ &\omega_f \left(\widetilde{\mathbf{U}} \right)^2 = \omega_f \left(\widetilde{\mathbf{V}}_f - \widetilde{\mathbf{V}}_s \right)^2. \end{aligned} \quad (18)$$

It is often assumed (see, e.g., [4], [27]) that $\left(\widetilde{\mathbf{U}} \right)^2 \approx \left(\widetilde{\mathbf{U}} \right)^2$ (or $\left(\widetilde{\mathbf{U}} \right)^2 \approx \left(\widetilde{\mathbf{U}} \right)^2$), and hence $\omega_f \approx 0$. Notice, such an assumption is not valid for laminar regimes, however, ω_f can be close to zero for the case of turbulent regime under sufficiently high Reynolds number (see Sec. 5).

Equating linear invariants of the tensors in (16) we obtain, in view of (18), the following estimation of the Reynolds stress tensor

$$-\overline{\rho_f} \widetilde{\mathbf{V}}_f'' \widetilde{\mathbf{V}}_f'' = \overline{\rho_f} \boldsymbol{\Pi}_f, \quad (19)$$

$$\mathbf{\Pi}_f = \mathbf{\Phi}_f - \frac{1}{3} \left(\omega_f \left| \widetilde{\mathbf{V}}_f - \widetilde{\mathbf{V}}_s \right|^2 + \text{tr} \mathbf{\Phi}_f \right) \mathbf{I}. \quad (20)$$

Using the Reynolds analogy (e.g., [3], [23]), as applied to multi-phase media, and following [26] and [32] we find, for the case of isotropic granular media, that the dispersivity tensor \mathbf{D}_f can be approximated as follows:

$$D_{f;ij} = F_1 \nu \delta_{ij} + F_2 \frac{l_*^2}{\nu} \widetilde{U}_i \widetilde{U}_j, \quad (21)$$

where

$$F_1 = \frac{a_f^0 Re^2}{1 + \alpha_f Re}, \quad F_2 = \frac{b_f}{1 + \beta_f Re}, \quad Re = \left| \widetilde{\mathbf{U}} \right| \frac{l_*}{\nu}, \quad (22)$$

a_f^0 , b_f , α_f , and β_f denote the dimensionless parameters describing the geometry of the void space, $\widetilde{U}_i = \widetilde{V}_{f;i} - \widetilde{V}_{s;i}$. We find, by virtue of (21)-(22), that

$$D_{f;ij} = \left(a_f l_* \left| \widetilde{\mathbf{U}} \right| \delta_{ij} + b_f l_* \widetilde{U}_i \widetilde{U}_j / \left| \widetilde{\mathbf{U}} \right| \right) \frac{Re}{1 + \beta_f Re}, \quad (23)$$

where $a_f = a_f^0 (1 + \beta_f Re) / (1 + \alpha_f Re)$. The similar approximation of the dispersivity tensor is suggested in [4, p. 218]. Notice, the approximation (23) of the dispersivity tensor \mathbf{D}_f does not contradict with Galilei-Newton principle of relativity [38, V. 2, p. 105], since \mathbf{D}_f depends on $\widetilde{\mathbf{U}} \equiv \widetilde{\mathbf{V}}_f - \widetilde{\mathbf{V}}_s$.

Similarly we obtain the macroscopic balance equations for the solid phase:

$$\frac{\partial}{\partial t} [(1 - \phi) \overline{\rho}_s] = -\nabla \cdot \left[(1 - \phi) \overline{\rho}_s \widetilde{\mathbf{V}}_s \right], \quad (24)$$

$$(1 - \phi) \overline{\rho}_s \frac{d}{dt} \widetilde{\mathbf{V}}_s = \nabla \cdot \overline{\boldsymbol{\sigma}}^f - (1 - \phi) \nabla \overline{p}_f + (1 - \phi) \overline{\rho}_s \widetilde{\mathbf{g}}_s + \mathbf{R} \cdot \left(\widetilde{\mathbf{V}}_f - \widetilde{\mathbf{V}}_s \right), \quad (25)$$

where $\overline{\boldsymbol{\sigma}}^f = (1 - \phi) (\overline{\boldsymbol{\sigma}}_s + \overline{p}_f \mathbf{I})$ is the Terzaghi effective stress [27]. Notice, following Nikolaevskiy [27], the Reynolds stress tensor (i.e., momentum dispersive flux [4]), $(1 - \phi) \overline{\rho}_s \widetilde{\mathbf{V}}_s'' \widetilde{\mathbf{V}}_s''$, is assumed as negligible for the solid phase. In such a case we obtain the conventional mathematical model for elastic-plastic deformations of granular media, i.e., the system of hyperbolic equations. However, as it can be concluded, e.g., from [4], the dispersion of the momentum should, in general, be taken into consideration if irreversible deformations take place. Then, in view of the Reynolds analogy, the momentum dispersive flux for the solid phase can be estimated by application of (19), (20), (15), and (23) with obvious modifications. In such a case we obtain the system of partial differential equations of parabolic type as a mathematical model of multi-phase dynamics.

3 Maximum principle

Hereinafter, the considered quantities, such as stress, density, etc., will be of the average ones only. Hence, the sign to indicate the fact of averaging will be deleted.

To develop a variational principle for plastic deformation of granular media we start with Coulomb friction law, as applied to a two-phase granular medium. According to Terzaghi's principle [27] the yield condition is formulated to the effective stresses:

$$C_\sigma \equiv \frac{2}{\sqrt{3}} |\sigma_\tau| + \alpha \sigma^f - Y = 0, \quad \sigma^f = \frac{1}{3} \text{tr} \boldsymbol{\sigma}^f, \quad (26)$$

where σ_τ denotes the shear stress intensity in the solid matrix, $\alpha (= \alpha(\chi) > 0)$ denotes the internal friction coefficient, χ denotes a hardening parameter, and $Y = Y(\chi)$ denotes the cohesion. Strain increment $d\boldsymbol{\varepsilon}$ of the matrix can be divided (e.g., [15], [27]) into elastic ($d\boldsymbol{\varepsilon}^e$) and plastic ($d\boldsymbol{\varepsilon}^p$) parts: $d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^e + d\boldsymbol{\varepsilon}^p$. The plastic strains are determined by non-associative flow rule [27] that can be written in the form

$$\mathbf{e}^p = \left[\boldsymbol{\sigma}^f + \frac{2}{3} \Lambda Y \mathbf{I} - \left(1 + \frac{2}{3} \Lambda \alpha \right) \sigma^f \mathbf{I} \right] \dot{\lambda}, \quad (27)$$

where $\Lambda = \Lambda(\chi)$ is the dilatancy rate, $\mathbf{e}^p \equiv d\boldsymbol{\varepsilon}^p/dt$, $\dot{\lambda} \equiv d\lambda/dt$. The scalar function $\dot{\lambda} = 0$ if $C_\sigma < 0$. Following Sedov [38, V. 4, pp. 145-147], in view of the first and second law of thermodynamics, as applied to the solid matrix of granular media, we obtain

$$(1 - \phi) \rho_s T \frac{d_i S}{dt} = - \frac{\mathbf{q} \cdot \nabla T}{T} + \boldsymbol{\theta} : \mathbf{e}^p \geq 0, \quad (28)$$

where $\boldsymbol{\theta} = \boldsymbol{\sigma}^f - (1 - \phi) \rho_s \partial F / \partial \boldsymbol{\varepsilon}^p$, S denotes the specific entropy, $d_i S$ denotes the specific entropy variation due to irreversible processes, T denotes the absolute temperature of the solid phase, $F \equiv U - TS = F(\boldsymbol{\varepsilon}^e, \boldsymbol{\varepsilon}^p, T)$ denotes the specific free energy, U denotes the internal energy, \mathbf{q} is the heat flux. Thus, the energy dissipation is determined by thermodynamic currents \mathbf{q} , \mathbf{e}^p and conjugated forces, according to the bilinear form of (28). In view of Curie principle [14], the value of energy dissipation associated with plastic deformation $D^p \equiv \boldsymbol{\theta} : \mathbf{e}^p \geq 0$. Hence, the non-associative flow rule (27) is bound to be equivalent to the following constitutive equation [14]: $\mathbf{e}^p = \mathbf{L} : \boldsymbol{\theta}$, where \mathbf{L} is a fourth-rank tensor. It is possible on condition that

$$\boldsymbol{\theta} = \begin{cases} \boldsymbol{\sigma}^f & \text{if } \Lambda = 0 \\ \boldsymbol{\sigma}^f - \frac{Y}{\alpha} \mathbf{I} & \text{if } \Lambda \neq 0 \end{cases}. \quad (29)$$

Thus, in the case of granular media, the specific free energy F is a function of the first invariant of plastic strain tensor $\boldsymbol{\varepsilon}^p$.

Following Sadovskii [34, Section 1.2], we assume that there exists a convex, possibly non-differentiable, function $B(\mathbf{e}^p)$ such that $\boldsymbol{\theta} \in \partial B(\mathbf{e}^p)$, where $\partial B(\mathbf{e}^p)$ denotes the sub-differential ([30], [34]) of the function B . It is assumed [34, Section 1.2] that $B(\mathbf{e}^p)$ is a positive-homogeneous function of \mathbf{e}^p , i.e., $B(b\mathbf{e}^p) = bB(\mathbf{e}^p)$ for $b \geq 0$. The above sub-differential relationship is equivalent ([30], [34]) to

$$\boldsymbol{\theta} : \mathbf{e}^p - B = \max_{\mathbf{e}_*^p} (\boldsymbol{\theta} : \mathbf{e}_*^p - B_*). \quad (30)$$

The right-hand side of (30) is the Young's transformation [34] of B , and, in view of the positive-homogeneousess, is the characteristic function of the convex set $\Upsilon = \{\boldsymbol{\theta} \mid C_\sigma \leq 0\}$, i.e. the right-hand side of (30) is equal to zero for $\boldsymbol{\theta} \in \Upsilon$, and hence $B = \boldsymbol{\theta} : \mathbf{e}^p$, i.e. $B \equiv D^p$. Using the Young's transformation for the characteristic function of the convex set Υ , we obtain

$$D^p = \max_{\boldsymbol{\theta}_* \in \Upsilon} (\boldsymbol{\theta}_* : \mathbf{e}^p). \quad (31)$$

In view of (31) we obtain the desired variational principle:

$$(\boldsymbol{\theta} - \boldsymbol{\theta}_*) : d \boldsymbol{\varepsilon}^p \geq 0, \quad \boldsymbol{\theta}, \boldsymbol{\theta}_* \in \Upsilon, \quad (32)$$

where $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_*$ denote the tensors associated with actual and, respectively, virtual energy dissipation.

4 Concluding remarks

To fulfill the basic laws of conservation by the mean quantities, the averages were introduced by different methods. In particular, the velocities and body forces are mass-averaged, whereas the densities and stresses are treated as volume averaged. Owing to this approach we succeeded in modeling of the Reynolds stress tensors under variable densities. Thus, the developed macroscopic balance equations are applicable for modeling flows through a permeable medium under a wide range of velocities.

The novel variational principle (32), developed for plastic deformation of granular media, can be interpreted as the maximum plastic energy dissipation principle. If a granular medium is idealized to be cohesionless ($Y = 0$), then in view of (29) $\boldsymbol{\theta} = \boldsymbol{\sigma}^f$, $\boldsymbol{\theta}_* = \boldsymbol{\sigma}_*^f$, and hence the novel variational principle (32) coincides with the maximum plastic work principle [15].

5 Appendix

Let us estimate the value of ω_f in (17) by considering the motion of an incompressible viscous fluid through a medium consisting of straight cylindrical tubes. Let us note that because of incompressibility we have $\tilde{e}_f \equiv \overline{e}_f$, $\forall e_f$.

First we consider Hagen-Poiseuille flow (see, e.g., [35, p. 117]), i.e., a steady-state laminar flow regime. In polar coordinates the velocity distribution over a cross-section of i -th tube can be written in the form

$$\mathbf{U}_i = \frac{\zeta_i R_i^2}{4\mu} \left(1 - \frac{r_i^2}{R_i^2}\right) \mathbf{k}_i \quad 0 \leq r_i \leq R_i, \quad (33)$$

where R_i denotes the radius of i -th tube, ζ_i denotes the pressure drop, and $\mathbf{k}_i \equiv (k_{x,i}, k_{y,i}, k_{z,i})$ denotes the unit vector parallel to the axis of the tube i .

By virtue of (33) we obtain that

$$\mathbf{J}_i \equiv 2\pi \int_0^{R_i} \mathbf{U}_i r_i dr_i = \frac{\pi \zeta_i R_i^4}{24\mu} \mathbf{k}_i, \quad (34)$$

Then clearly

$$\overline{\mathbf{U}} = \frac{\sum \mathbf{J}_i}{\sum \pi R_i^2} = \frac{\sum \zeta_i R_i^4}{8\mu \sum R_i^2} \mathbf{k}_i. \quad (35)$$

Let $\xi_i = \zeta_i R_i^4$. Then, in view of (35), we find

$$(\overline{\mathbf{U}})^2 = \frac{(\sum \xi_i k_{x,i})^2 + (\sum \xi_i k_{y,i})^2 + (\sum \xi_i k_{z,i})^2}{64\mu^2 (\sum R_i^2)^2}. \quad (36)$$

By virtue of the elementary inequality, $a^2 + b^2 \geq 2ab$, it is an easy matter to prove that the numerator in the right-hand side of (36) is bounded above by $(\sum \xi_i)^2$. Hence we get

$$(\overline{\mathbf{U}})^2 \leq \frac{(\sum \zeta_i R_i^4)^2}{64\mu^2 (\sum R_i^2)^2}. \quad (37)$$

In a similar manner, we obtain

$$J_i^* \equiv 2\pi \int_0^{R_i} (\mathbf{U}_i)^2 r_i dr_i = \frac{\pi \zeta_i^2 R_i^6}{48\mu^2}. \quad (38)$$

Then obviously

$$\overline{(\mathbf{U})^2} = \frac{\sum J_i^*}{\sum \pi R_i^2} = \frac{\sum \zeta_i^2 R_i^6}{48\mu^2 \sum R_i^2}. \quad (39)$$

In view of (39) and (37) we obtain from (17) that

$$\omega_f \geq \frac{4 \sum R_i^2 \sum \zeta_i^2 R_i^6}{3 (\sum \zeta_i R_i^4)^2} - 1. \quad (40)$$

The lower bound of ω_f follows from Cauchy-Schwarz inequality

$$\left(\sum a_i b_i \right)^2 \leq \sum a_i^2 \sum b_i^2. \quad (41)$$

Setting $a_i = R_i$ and $b_i = \zeta_i R_i^3$, we obtain from (40), in view of (41), that $\omega_f \geq 1/3$. Notice, $\omega_f = 1/3$ if the tubes are identical in diameter and their axes are parallel to each other.

Next we consider a turbulent regime. The velocity distribution over a cross-section of i -th tube can be written (e.g., [36, p. 563]) in the following form

$$\mathbf{U}_i = U_{m,i} \left(1 - \frac{r_i}{R_i} \right)^{1/n} \mathbf{k}_i \quad 0 \leq r_i \leq R_i, \quad n = \text{const}, \quad (42)$$

where $U_{m,i}$ denotes the maximum velocity. Notice, we assume $n = \text{const}$ in the empirical formula (42), as the exponent n varies slightly with the Reynolds number Re [36, p. 563]. In the perfect analogy to the deduction of (37), (39), and (40), we obtain in the case of turbulent regime that

$$(\overline{\mathbf{U}})^2 \leq \frac{4n^4 (\sum U_{m,i} R_i^2)^2}{(n+1)^2 (2n+1)^2 (\sum R_i^2)^2}, \quad \overline{(\mathbf{U})^2} = \frac{n^2 \sum U_{m,i}^2 R_i^2}{(n+1)(n+2) \sum R_i^2}, \quad (43)$$

$$\omega_f \geq \frac{(2n+1)^2 (n+1) (\sum R_i^2) \sum U_{m,i}^2 R_i^2}{4n^2 (n+2) (\sum U_{m,i} R_i^2)^2} - 1. \quad (44)$$

Let $a_i = R_i$ and $b_i = U_{m,i} R_i^2$ in (41). Then, in view of (44), we obtain

$$\omega_f \geq \frac{5n+1}{4n^2 (n+2)}. \quad (45)$$

In particular, if $n = 6$, i.e. $Re = 4 \cdot 10^3$ [36, p. 563], then we find, by virtue of (45), that $\omega_f \geq 0.027$. Thus, in the case of turbulent regime the minimal value of ω_f is far less than in the case of laminar regime.

References

- [1] M. Alam and S. Luding, Rheology of bidisperse granular mixtures via event-driven simulations, *J. Fluid Mech.*, V. 476 (2003), 69-103.
- [2] M. Alam, V. H. Arakeri, P. R. Nott, J. D. Goddard and H. J. Herrmann, Instability-induced ordering, universal unfolding and the role of gravity in granular Couette flow, *J. Fluid Mech.*, V. 523 (2005), 277-306.
- [3] D. A. Anderson, J. C. Tannehill, and R. H. Pletcher, *Computational fluid mechanics and heat transfer*, Hemisphere Publishing Corporation, New York, 1984.
- [4] J. Bear and Y. Bachmat, *Introduction to modeling of transport phenomena in porous media*, Kluwer Acad. Publishers, 1990.
- [5] J. Bear and A. Verruijt, *Modeling Groundwater Flow and Pollution*, Published Dordrecht : D. Reidel, 1987
- [6] S. M. Belotserkovsky and A. S. Ginevsky, *Turbulent jets and wakes simulation using the method of discrete vortices*, "Physico-Matematicheskaya literatura", Moscow, 1995 (in Russian).
- [7] L. S. Bennethum and T. Giorgi, Generalized Forchheimer Equation for Two-Phase Flow Based on Hybrid Mixture Theory, *Transport in Porous Media* 26: 261-275, 1997.

- [8] V. S. Borisov and S. Sorek, On monotonicity of difference schemes for computational physics, SIAM Journal on Scientific Computing, Vol. 25, No. 5, (2004), 1557-1584.
- [9] V. S. Borisov, M. Mond, On monotonicity, stability, and construction of central schemes for hyperbolic conservation laws with source terms, (2007), arXiv:0705.1109v1 [physics.comp-ph].
- [10] E. C. Chan and F.-S. Lien, Permeability Effects of Turbulent Flow Through a Porous Insert in a Backward-Facing-Step Channel, Transport in Porous Media, Vol. 59, No. 1, (2005), 47 - 71.
- [11] Gedeon Dagan, Flow and transport in porous formations, Springer-Verlag, Berlin, 1989.
- [12] A. K. Didwania and R. de Boer, Saturated Compressible and Incompressible Porous Solids: Macro- and Micromechanical Approaches, TiPM, V. 34: 101-115, 1999.
- [13] P. G. Frick, *Turbulence: aproaches and models*, Institute of computer science, Moscow-Izhevsk, 2003 (in Russian).
- [14] I. Gyarmati, *Non-equilibrium thermodynamics, field theory and variational principles*, Springer-Verlag, Berlin, 1970.
- [15] W. Han and B. D. Reddy, *Plasticity, mathematical theory and numerical analysis*, Springer-Verlag, New York, 1999.
- [16] S. Jagering, R. de Boer, and S. Breuer, Elastoplastic compaction of metallic powders, International Journal of Mechanical Sciences 43 (2001), 1563-1578.
- [17] Y. Jiang and M Liu, Granular elasticity without the Coulomb condition, Phys. Rev. Lett. V. 91, 144-301 (2003).
- [18] Gabriel G Katul, Larry Mahrt, Davide Poggi, and Christophe Sanz, One- and two-equation models for canopy turbulence, Boundary-Layer Meteorology 113: 81-109, 2004.
- [19] Nikoly I. Kolev, *Multiphase flow dynamics 1, Fundamentals*, Springer-Verlag, Berlin, 2005
- [20] Nikoly I. Kolev, *Multiphase flow dynamics 2, Mechanical and thermal interactions*, Springer-Verlag, Berlin, 2005
- [21] J. L. Lage, M. J. S. de Lemos and D. A. Nield, Modeling turbulence in porous media, in: D. Ingham, I. Pop (Eds.), Transport Phenomena in Porous Media II, Pergamon Press, Amsterdam, 2002, pp. 198-230 (Chapter 8).

- [22] Marcelo J. S. de Lemos and Maximilian S. Mesquita, Turbulent mass transport in saturated rigid porous media, *Int. Comm. Heat Mass transfer*, 30(1), 105-113, 2003.
- [23] A. S. Monin and A. M. Yaglom, *Statistical fluid mechanics: mechanics of turbulence*, V. 1, The Mit Press, Cambridge, Massachusetts, and London, England, 1971.
- [24] A. Nakayama and F. Kuwahara, A Macroscopic Turbulence Model for Flow in a Porous Medium, *Journal of Fluids Engineering*, V. 121, 437-433, 1999.
- [25] R. M. Nedderman, *Statics and kinematics of granular materials*, Cambridge University Press, 1992.
- [26] V.N. Nikolaevskii, Convective Diffusiobn in Porous Media, *PMM* V. 23, No. 6, pp. 1042-1050, 1959.
- [27] V.N. Nikolaevskij, *Mechanics of porous and fractured media*, World Scientific, Singapure, 1990.
- [28] V. N. Nikolaevskiy, *Geomechanics and fluidodynamics*, Kluwer Academic Publishers, Dordrecht, 1996.
- [29] Ya. Pan and R. N. Horne, Generalized Macroscopic Models for Fluid Flow in Deformable Porous Media I: Theories, *TiPM*, V. 45, pp. 127, 2001.
- [30] P. D. Panagiotopoulos, *Inequality problems in mechanics and applications*, Birkhäuser, Boston, 1985.
- [31] Marcos H. J Pedras and Marcelo J. S. de Lemos, Computation of turbulent flow in porous media using a low-Reynolds k-e model and an infinite array of transversally displaced elliptic rods, *Numerical Heat Transfer, Part A*, 43: 585-602, 2003.
- [32] M. Poreh, The dispersivity tensor in isotropic and axisymmetric mediums, *J. Geophys. Res.*, V. 7, No 16 (1965), 3909-3913.
- [33] M. Prat, F. Plouraboue, and N. Letalleur, Averaged Reynolds Equation for Flows between Rough Surfaces in Sliding Motion, *Transport in Porous Media* 48: 291-313, 2002.
- [34] V. M. Sadvskii, *Discontinuous solutions in dynamic elastic-plastic problems*, Physical and Mathematical Literature Publishing Company, Russian Academy of Sciences, Moscow, 1997 (in Russian).
- [35] Herrmann Schlichting and Klaus Gersten, *Boundary-layer theory*, Springer-Verlag, Berlin, 2000.
- [36] Herrmann Schlichting, *Boundary-layer theory*, Sixth Edition, McGraw-Hill, New York, 1968.

- [37] Skjetne E. and Auriault J.-L., High-Velocity Laminar and Turbulent Flow in Porous Media *TiPM* 36: 131-147, 1999.
- [38] L. I. Sedov, *A course in continuum mechanics*, Wolters-Noordhoff Publishing, Groningen, the Netherlands, 1971.
- [39] Peter Vadasz and Shmuel Olek, Weak Turbulence and Chaos for Low Prandtl Number Gravity Driven Convection in Porous Media, *Transport in Porous Media* 37: 69-91, 1999.
- [40] X. Y. Zhou and J. C. F. Pereira, A Multidimensional Model for Simulating Vegetation Fire Spread using a Porous Media Sub-model, *Fire. Mater.* 24, 37-43 (2000)